PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

# Calculation of Supersonic Flows with Heat Addition

E. G. Broadbent

*Phil. Trans. R. Soc. Lond. A* 1991 **335**, 153-160
doi: 10.1098/rsta.1991.0041

# Calculation of supersonic flows with heat addition

By E. G. Broadbent
*Imperial College of Science, Technology and Medicine, London SW7 2BZ, U.K.*

In the 1970s a method was published, and applied to various configurations, for calculating two-dimensional steady inviscid supersonic flow fields with heat addition. The method is of inverse type in that for a given geometry the distribution of heat added to the flow is not prescribed in advance; instead the streamlines are prescribed and the heat addition necessary to satisfy the Euler equations is then found. The present paper describes how it can be adapted to satisfy the Euler equations for a given mode of heat addition. Its main advantage is that the finite difference technique is always applied along the mathematical characteristics of the equations being satisfied at each stage of the calculation.

## 1. Introduction

It is well known that supersonic combustion can be used to generate pressure fields that will provide a propulsive force for a hypersonic aircraft. The combustion process itself is complicated and here we consider only the effects of heat (and mass) addition on the flowfield. Even so, mathematical analysis is complicated by the fact that heat addition produces changes in entropy, stagnation pressure and Mach number (as well as introducing vorticity) so that the characteristics of the Euler equations can undergo rapid changes of direction.

Oswatitsch (1959) used linear theory to show how heat addition to the flow adjacent to the downstream face of a double wedge could provide a propulsive force and Zierep (1966) extended this work to larger wedge angle using the hypersonic approximation. I adopted an inverse method to analyse more realistic geometries (see, for example, Broadbent 1971, 1973, 1976) and this provides the basis of the method described here, although it is no longer restricted to the inverse approach. Much of the early work is described by Küchemann (1978).

The method has been applied to two-dimensional (steady) flow (plane or axisymmetric) with heat addition, and in its original form the streamlines are chosen, so that $\theta$, the flow direction, is everywhere specified. The four governing equations (continuity, two components of momentum and energy) are then solved for the four field variables pressure $p$, flow speed $u$, density $\rho$ and heat addition per unit volume per second $Q$. Choice of streamlines may seem unwarranted, but the method was intended to provide a rapid way of comparing lift and net wave drag on a range of section shapes with external or ducted heat addition, and this it does very effectively. Bounding streamlines are known for each configuration and the method allows the pressure to be specified along one wall, which is just what is wanted for this kind of calculation; streamline shape can be specified algebraically with adjustable parameters so that the effect of different modes of heat addition can be examined.

*Phil. Trans. R. Soc. Lond.* A (1991) **335**, 153–160
*Printed in Great Britain*

7-2

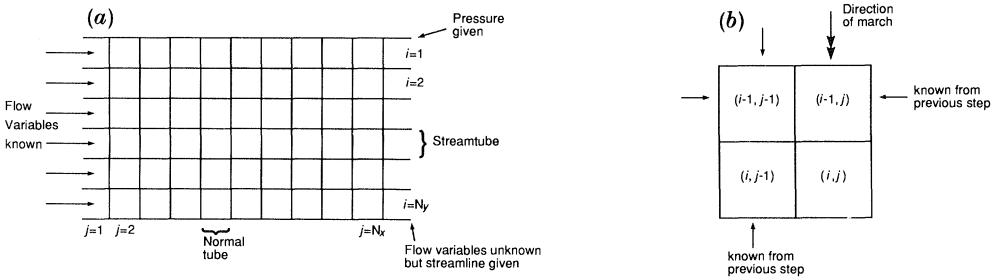


Figure 1. Mesh diagram (a) general arrangement; (b) local mesh for solution of momentum equations for the element $(i,j)$.

The four equations may be written, in streamwise coordinates for plane two-dimensional steady inviscid flow

$$\partial(\rho u w)/\partial s = m_{\mathrm{f}}\, w, \tag{1}$$

$$\rho u\, \partial u/\partial s = -\partial p/\partial s - m_{\mathrm{f}}\, u, \tag{2}$$

$$\rho u^2\, \partial\theta/\partial s = -\partial p/\partial n, \tag{3}$$

$$\frac{\partial}{\partial s}\left(\frac{\gamma}{\gamma-1}\, puw\right) + \frac{1}{2}\frac{\partial}{\partial s}(\rho u^3 w) = m_{\mathrm{f}}\, w C_{v_{\mathrm{f}}} T_{\mathrm{f}} + Qw, \tag{4}$$

where $\rho$ is density, $u$ is flow speed, $w$ is width of an incremental stream tube, $m_{\mathrm{f}}$ is rate of addition of fuel mass per unit volume, $p$ is pressure, $\theta$ is flow direction, $\gamma$ is ratio of specific heats, $C_{v_{\mathrm{f}}}$ is specific heat at constant volume of the added fuel, $T_{\mathrm{f}}$ is absolute temperature of the added fuel, $Q$ is heat added to the flow per unit volume (e.g. by combustion) and ($\mathrm{d}s$, $\mathrm{d}n$) are intrinsic coordinates along the streamlines and normals respectively.

The terms in $m_{\mathrm{f}}$ were omitted in the earlier work but are included here because they can be significant, even for hydrogen, at high supersonic Mach numbers. In the analysis given here, $m_{\mathrm{f}}$ is presumed to be initially at rest relative to the vehicle, although allowance could be made for any known form of injection.

The method of solution was first to assume that $\theta$ was known throughout the flowfield and to set up a mesh along the streamlines and normals. Then with $m_{\mathrm{f}} = 0$ (or alternatively with $m_{\mathrm{f}}$ a given function of position) and $w$ determined by the distribution of $\theta$, equation (1) determines $\rho u$ as a function of position, whence substitution in (2) and (3) leads to a pair of linear hyperbolic equations in $\rho$ and $u$ with real characteristics that lie along the streamlines and normals. Numerical solution via difference equations is then straightforward and very fast. If $p(i,j)$, $u(i,j)$ denote the pressure and flow speed at the intersection of stream tube $i$ and normal tube $j$ (see figure 1), then (2) and (3) give a pair of linear equations to solve for $p(i,j)$, $u(i,j)$ in terms of the values at $(i-1,j)$ and $(i,j-1)$. Thus with upstream conditions given ($j = 1$) together with the pressure along one bounding streamtube ($i = 1$), $p$ and $u$ can be determined throughout the field by marching along successive normals away from $i = 1$ and $j = 1$. Finally, since $p$ and $\rho$ are now known, $T$ and hence $\gamma$ can be found from an equation of state and then (4) determines $Q$, the heat addition necessary to satisfy the chosen streamlines and bounding pressure distribution. Broadbent (1973) gives a wide range of results obtained in this way for hypersonic sections designed to give a net propulsive force by means of direct heat addition to the supersonic flow. Values of the sectional lift and wave-drag coefficients quoted are in the ranges 0.03 to 0.06 and $-0.03$ to $-0.07$ respectively for a free-stream Mach number of 7.5.

Although the method has many attractive features it also has disadvantages. It is sometimes difficult to design suitable streamlines, and small regions of negative heat addition are often a bugbear. This inverse nature of the solution, where $Q$ appears as the outcome of the calculation rather than the input to it can, however, be avoided as described in §2 which also gives a brief illustration of the potential importance of mass addition. Some examples for given $Q$ distribution are given in §3 and more recent developments described in §4.

## 2. Solution for given heat and mass addition

The solution of (1)–(4) follows the same basic pattern as that outlined in §1 except that in this case $\theta$ (and $w$) are unknown and must be found by an iterative method starting from an initial trial. For clarity, consider a free supersonic plane two-dimensional heated jet, symmetric about its centreline which can be taken to be the $x$ axis. If $N_y$ streamtubes are to be used in the mesh, these are defined by $N_y$ streamlines together with the axis of symmetry. For the next normal downstream of known conditions let

$$\boldsymbol{\theta} = \boldsymbol{\theta}_{\mathrm{u}} + \boldsymbol{z}, \tag{5}$$

where $\boldsymbol{\theta}$ is an $N_y$-element vector containing the streamline slopes, $\boldsymbol{\theta}_{\mathrm{u}}$ is the known vector of $\theta$ for the normal immediately upstream and $\boldsymbol{z}$ is an unknown vector that may initially be taken as zero. The correct solution for $\boldsymbol{z}$ is then found by Newton's method, for given vectors $\boldsymbol{Q}$ and $\boldsymbol{m}_{\mathrm{f}}$.

For example, let

$$\boldsymbol{F} = \boldsymbol{Q} - \boldsymbol{Q}_0, \tag{6}$$

where $\boldsymbol{Q}_0$ is the required mode of heat addition. Then $\boldsymbol{Q}$ can be calculated for the initial $\boldsymbol{z}$ using the method of §1, provided the bounding pressure is known at one end of the normal. In this example the pressure at the edge of the jet can be calculated by standard supersonic theory applied to the turning of the free stream according to the bounding $\theta$ given by (5). Thus a march along the normal towards the axis of symmetry is possible, and the resulting vector $\boldsymbol{F}$ is determined. An incremental change to $z(1)$ then allows the vector $\partial\boldsymbol{F}/\partial z(1)$ to be calculated numerically, and similarly $\partial\boldsymbol{F}/\partial z(i)$ for $i = 1, \ldots, N_y$ which completes the jacobian $A$ say, where

$$A \equiv \begin{pmatrix} \partial F(1)/\partial z(1) \ldots \partial F(1)/\partial z(N_y) \\ \vdots \qquad\qquad \vdots \\ \partial F(N_y)/\partial z(1) \ldots \partial F(N_y)/\partial z(N_y) \end{pmatrix}. \tag{7}$$

Newton's method then gives

$$A \cdot (\boldsymbol{z}^{(k+1)} - \boldsymbol{z}^{(k)}) = -\boldsymbol{F}(\boldsymbol{z}^{(k)}), \tag{8}$$

where the new vector, $\boldsymbol{z}^{(k+1)}$, is an improvement on the previous choice $\boldsymbol{z}^{(k)}$, and is readily determined by solution of the set of linear equations (8).

If the flow is axisymmetric, the only change to the governing equations (1)–(4) is that $w$ is replaced by $wy$ throughout, where $y$ is the (mean) distance of the stream-tube element from the axis of symmetry, since $wy$ is proportional to the annular area of the element. For an axisymmetric free jet, the calculation of the bounding pressure from the supersonic free stream is less straightforward, but an alternative possibility is to increase $N_y$ so as to include a number of unheated streamtubes outside the jet. If the outermost of these lies outside the Mach cones of the pressure waves from the jet it will be at the ambient pressure along the length of interest, and the solution can proceed as before. Naturally, for these external tubes $Q_0 = 0$.

A slight variation is needed to analyse a ducted flow because it is no longer possible to calculate the pressure along either bounding streamline. On the other hand, for $N_y$ streamtubes there are only $N_y - 1$ unknown values of $\theta$ (since $\theta$ is known for each bounding streamline) and the final unknown may thus be a bounding value of pressure (or flow speed), e.g.

$$\{z(1), \ldots, z(N_y-1), z(N_y)\} = \{\theta(1)-\theta_u(1), \ldots, \theta(N_y-1)-\theta_u(N_y-1), p(N_y)-p_u(N_y)\}. \tag{9}$$

Other variations are open to choice. For example the unknown vector $\boldsymbol{z}$ can be allocated to any set of parameters that enable a complete normal of the mesh to be constructed;

$$\boldsymbol{w} = (1+z)\,\boldsymbol{w}_\mathrm{u} \tag{10}$$

has been used where $z$, here, is a diagonal matrix. Nor is it necessary to use Newton's method, involving the time-consuming construction of the jacobian; e.g. a quasi-Newton method could be used instead. Moreover convergence can be problem with Newton's method if the initial choice for $\boldsymbol{z}$ is too remote from the solution. In some applications to more complicated flowfields with strong gradients it was necessary to adjust the geometry, or the amount of heat, in stages. This was done by solving for a geometry with smaller gradients, and if $N_x$ is the number of normals covering the flowfield by then storing a matrix containing the $N_x$ converged vectors $\boldsymbol{z}$; next the geometry was moved a step nearer to the required geometry and the converged sets of $\boldsymbol{z}$ were used as the new trial values. The whole sequence of steps was readily included in the program. Such an approach is time-consuming, but on the other hand when Newton's method does converge it does so very rapidly.

The numerical examples have all been run on a Zenith PC with a FORTRAN compiler and the computing time for a complete run in a recent example was 35 s. This particular example was for supersonic duct flow with a two to one expansion ratio and with heat addition. The program was in FORTRAN double precision with $N_y = 7$, $N_x = 20$ and the accuracy required for solution of the linear equations was $10^{-9}$ and for convergence in the Newton iterations was $10^{-7}$. The initial choice for each normal was $\boldsymbol{z} = 0$ and the number of iterations was either 3 or 4 for all normals.

In all the examples given in this paper the mass addition $m_\mathrm{f}$ has been neglected for simplicity, although it has been included in some practical applications. The reason for its potential importance at higher supersonic Mach numbers can be illustrated quite simply. Elementary analysis for quasi one-dimensional steady inviscid flow in the $x$ direction gives

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\gamma M^2}{M^2-1}\left\{-\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\gamma-1}{\gamma}\frac{Q}{pu} + \frac{m_\mathrm{f}}{\rho u}\left(\frac{\rho C_{v_\mathrm{f}} T_\mathrm{f}}{p}\frac{(\gamma-1)}{\gamma} + 1 + \frac{\gamma-1}{2}M^2\right)\right\}, \tag{11}$$

where $M$ is the Mach number, and $A$ is the cross-sectional area of the duct.

Suppose that for a given distribution of $Q$ and $m_\mathrm{f}$ the area distribution is adjusted





PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

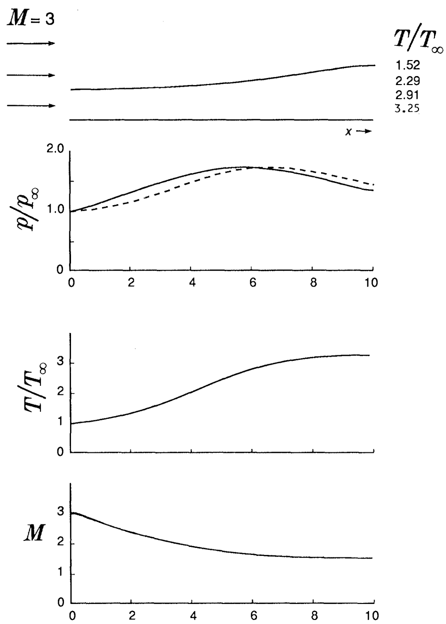


Figure 2. Two-dimensional symmetric free jet for $M_\infty = 3$. The top diagram shows the geometry of the upper half of the jet with downstream temperatures. The second diagram shows the pressure distribution for the innermost stream tube (full line) and along the jet periphery (dashed line); below are the temperature and Mach number distributions for the innermost stream tube. The energy balance is: in 59.102, out 59.090.

to keep $p$ constant. Then if $A$ increases from $A_1$ to $A_2$, the thrust will be given by $p(A_2 - A_1)$, where the area increase is given by integration of

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\gamma-1}{\gamma}\frac{Q}{pu} + \frac{m_\mathrm{f}}{\rho u}\left\{\frac{\rho C_{v_\mathrm{f}} T_\mathrm{f}}{p}\frac{(\gamma-1)}{\gamma} + 1 + \frac{\gamma-1}{2}M^2\right\}. \tag{12}$$

Clearly the relative importance of $m_\mathrm{f}$ to $Q$ increases like $M^2$ at large $M$. The derivation of (11) assumes that a quantity $m_\mathrm{f}(x)$ is smoothly accelerated to the speed $u(x)$ at position $x$. In a practical application the problem of mixing requires detailed consideration.

## 3. Simple examples

Figure 2 shows results for a symmetric two-dimensional free jet in a uniform supersonic flow at $M = 3$. The distribution of heat addition chosen was

$$Q_\mathrm{c} = Q_{\mathrm{c}0} \cos \tfrac{1}{2}\pi\eta \sin \pi\xi, \tag{13}$$

where $Q_\mathrm{c}$ is the heat supply per cell of the mesh, $Q_{\mathrm{c}0}$ is an adjustable constant, $\eta$ is a normal coordinate running from 0 on the axial streamline to 1 on the peripheral streamline, and $\xi$ is a streamwise coordinate running from 0 at the upstream end to 1 at the downstream end. The inlet Mach number and temperature were taken to be 3 and 1000 K respectively, these being typical of the flow downstream of a hypersonic intake. For the thermodynamics the gas constant $R$ was kept constant but $\gamma$ was allowed to vary with temperature according to

$$\gamma = 1.4 - 0.12(1 - \mathrm{e}^{-(T/1200)^2}). \tag{14}$$

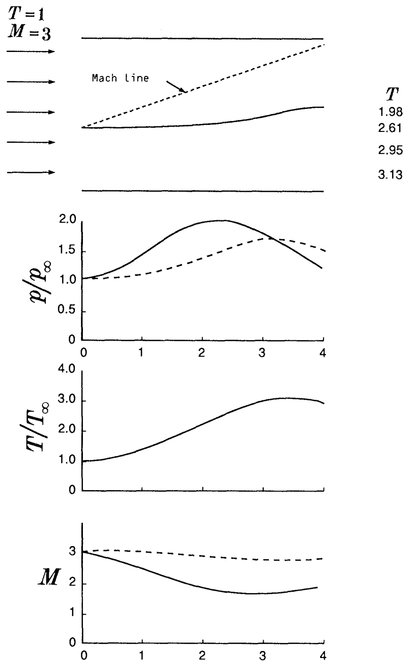


Figure 3. Axisymmetric free jet ($M_\infty = 3$). The top diagram shows the jet periphery and the outermost streamline of the calculation. In the lower diagrams distributions are given for the innermost stream tube (full line) and innermost unheated stream tube (dashed line). The energy balance is: in 120.197, out 120.241.

Table 1

| | $T/T_\infty$ | $p/p_\infty$ | $u/u_\infty$ | $\rho u/\rho_\infty u_\infty$ |
|---|---|---|---|---|
| innermost stream tube | 3.474, 3.470 | 1.954, 1.949 | 3.264, 3.271 | 1.836, 1.837 |
| outermost stream tube | 1.645, 1.650 | 2.223, 2.256 | 3.176, 3.168 | 4.291, 4.330 |

which agrees fairly well with experimental values for air up to about 3000 K. $Q_{c0}$ was chosen to give a peak temperature of just over 3000 K. This calculation was in single precision with $N_y = 4$, $N_x = 10$, but even for such a coarse mesh the energy balance is quite good: the energy in comprises the upstream rate of flow of energy into the jet plus the total rate of heat supply, and the energy out comprises the downstream rate of flow of energy out of the jet.

In this example, the peripheral pressure of the jet was calculated from the turning of the free stream, but as noted in §2 for axisymmetric flow it may be simpler to include sufficient external stream tubes to reach the parallel free stream at $p_\infty$. The effectiveness of this can easily be checked for a plane jet by comparing the two methods. This has been done for a jet like that of figure 2 but shorter and results are indicated in table 1.

In this table the first number relates to the calculation with $N_y = 4$ and $N_x = 10$, the second has $N_y = 10$, $N_x = 10$ where only stream tubes 1–4 are heated. The second calculation was in double precision because of the larger set of linear equations.

Figure 3 shows a similar calculation for an axisymmetric free jet using four heated and six unheated stream tubes. Again the jet is shorter than in figure 2 (a longer jet

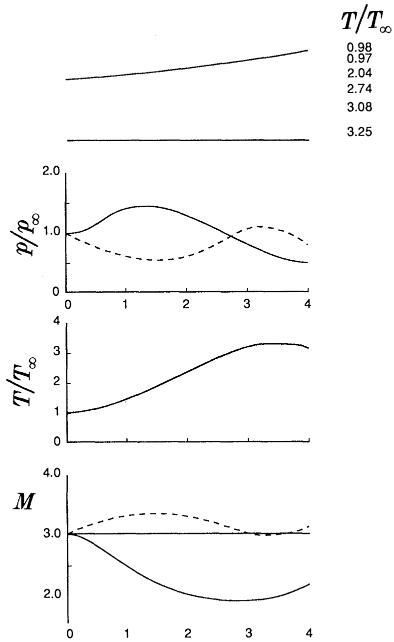


Figure 4. Axisymmetric duct flow with four heated and two unheated stream tubes. In the lower diagrams the full line is for the innermost and the dashed line for the outermost stream tube. The energy balance is: in 24.083, out 24.128.

would have needed more unheated stream tubes) and the resulting pressure distribution is of some interest. The innermost stream tube is widening so fast at the downstream end that the temperature actually starts to fall despite the continuing heat addition. In this case and the next, the cosine term in (13) is square-rooted, so that the radial cut-off of heat is sharper.

Figure 4 shows an axisymmetric duct flow with $N_y = 6$, the four innermost stream tubes being heated and the two outermost unheated. In a flow such as this the interactions are considerable as the pressure distribution shows.

It is perhaps rather remarkable that a solution can be found to supersonic flows with complicated wave interactions without taking direct account of the waves. What happens is that the calculation treats a hypothetical gas constrained to follow fixed streamlines by suitable release of heat, and for this hypothetical gas the characteristics follow the streamlines and normals. Newton's method then adjusts the streamlines until they finally coincide with those of a real Euler gas. It is the fact that the finite differences are always taken along the mathematical characteristics that provides the principal advantage of the method. Application of the method to a more realistic hypersonic configuration is illustrated by Townend (this Theme).

## 4. Developments

It was noted in §2 that convergence is not always easy to achieve. This can be improved and the accuracy increased at the same time by not constructing the mesh in the physical plane. Instead one transforms from coordinates $(x, y)$ to coordinates $(q_1, q_2)$ say which follow the streamlines and normals (Broadbent 1971, Appendix) with

$$ds = h_1 \, dq_1 \quad dn = h_2 \, dq_2 \tag{15}$$





PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and
$$\partial h_2/\partial q_1 = h_1\,\partial\theta/\partial q_2, \quad \partial h_1/\partial q_2 = -h_2\,\partial\theta/\partial q_1, \tag{16}$$

where $h_1$ and $h_2$ are the usual scale parameters with dimensions of length and $q_1$ and $q_2$ are non-dimensional orthogonal coordinates (they are equivalent to $\xi, \eta$ in (13)). In the $(q_1, q_2)$ plane the mesh becomes a simple rectangular grid, so the step of constructing a normal in the physical plane is replaced by finite difference solution of (16) along a line $q_2 = \text{const}$. With $h_1, h_2$ and $\theta$ known on the $(q_1, q_2)$ grid the geometry is specified and $(x, y)$ can be obtained for each point by numerical summation. The main advantage lies in the fact that $h_1, h_2$ and $\theta$ are assumed to vary linearly from grid point to grid point (in the finite difference solution of (16)) so that $\theta$ is everywhere continuous, whereas it changes discontinuously at the nodes of the mesh described in §2. In other respects the solution can proceed as before; the computing times quoted in §2 were for an example using this technique.

## References


Broadbent, E. G. 1971 An exact numerical method of calculating inviscid heated flows in two dimensions with an example of duct flow. *Ing. Arch.* **40**, 14–28.

Broadbent, E. G. 1973 Numerical evaluation of further supersonic sections including ducted heat addition and nozzle flow. *Ing. Arch.* **43**, 395–415.

Broadbent, E. G. 1976 Flows with heat addition. *Prog. Aerospace Sci.* **17**, 93–108.

Küchemann, D. 1978 *The aerodynamic design of aircraft*. Oxford: Pergamon.

Oswatitsch, K. 1959 Antriebe mit Hechizung bei Uberschallgeschwindigkeit. *DVL Bericht* 90.

Zierep, J. 1966 Über den Einfluss der Warmezufuhr bei Hypershallströmungen. *Acta Mech.* **2**, 217–230.